From Fourier to the Heisenberg Uncertainty Principle

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Motivation

- Remember these guys?

Motivation

Lecture 1:

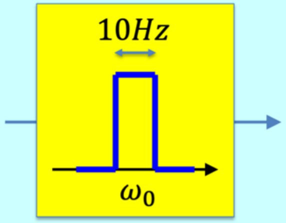
Brain Teaser #1

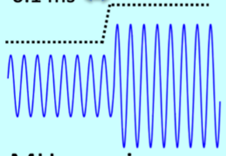
- A salesman from BrightLight.com is trying to sell you a new laser.
- He says it can produce extremely short pulses of light (2 fs) that are also very monochromatic in the green region of the spectrum (λ = 0.5 μm).
- Do you think his claim is plausible?
  - Yes, such a laser could be built with the right technology development
  - No, the optical signal he's describing violates a fundamental limit

Lecture 17:

ted New Bandpass Filter for AM Radio

Bandpass filter you can tune to your favorite station!





Narrow bandwidth of 10Hz to filter out noise and interference! Rapid response to changes in audio signal level 0.1 ms to pass through high pitches and bright timbre! Do you believe his claims? Why or why not? (Claims are wrong.)

Motivation

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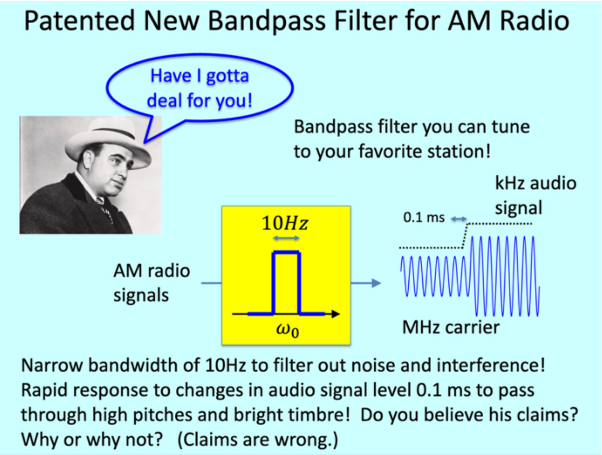


For variables related by the Fourier transform, like:

- Frequency and Time
- Frequency range and Temporal range

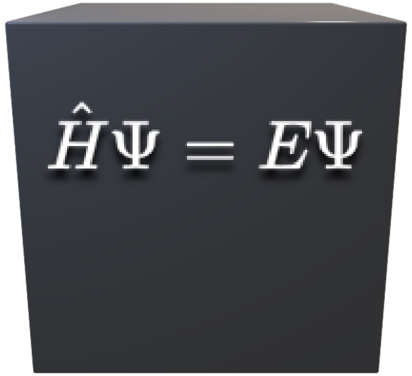
The precision of one variable fundamentally constrains the precision of the other.

Quantum Mechanics

- But Wait! Quantum mechanics says that certain properties of matter, like position and momentum, are related by the Fourier transform!
- You don’t know Quantum mechanics?

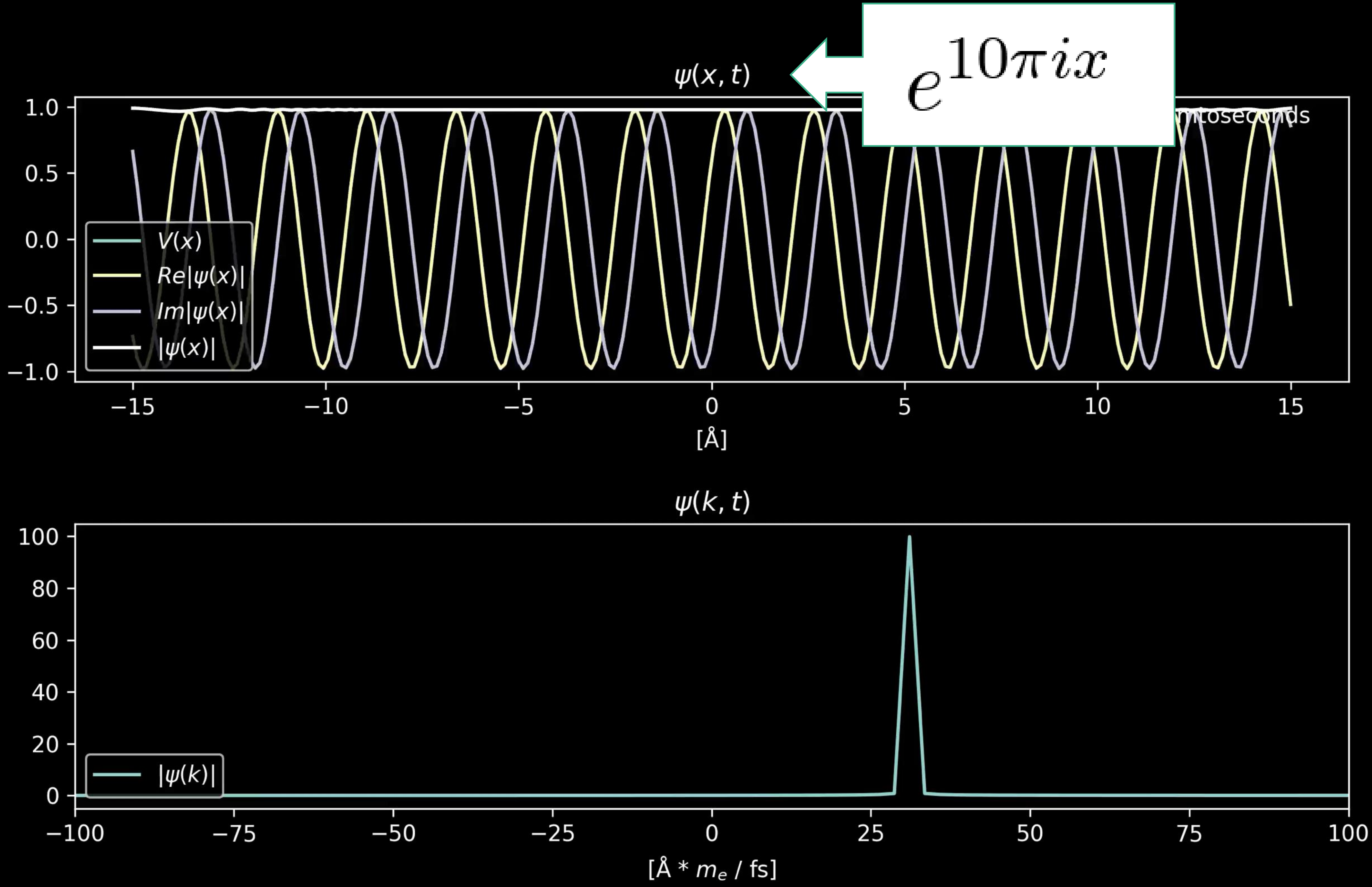
Quantum Mechanics TLDR

- Particles are more accurately described as waves.
  - Specifically, a function of position and time called the wavefunction Ψ.
- The wavefunction must evolve according to the Schrodinger equation
  - Can be derived from the potential & kinetic energy formula but we'll treat it as a black box

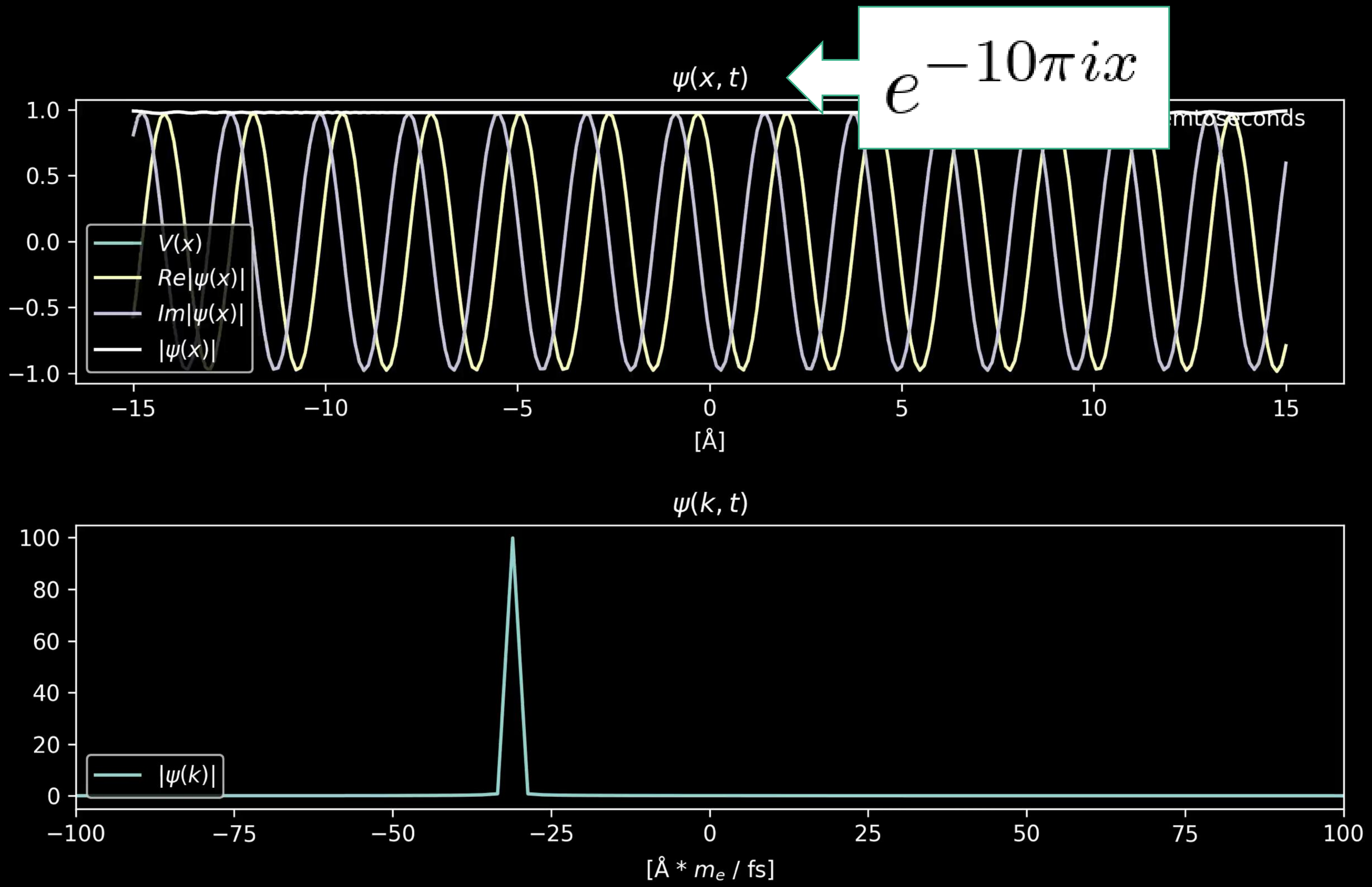


But why is momentum the Fourier transform of position?

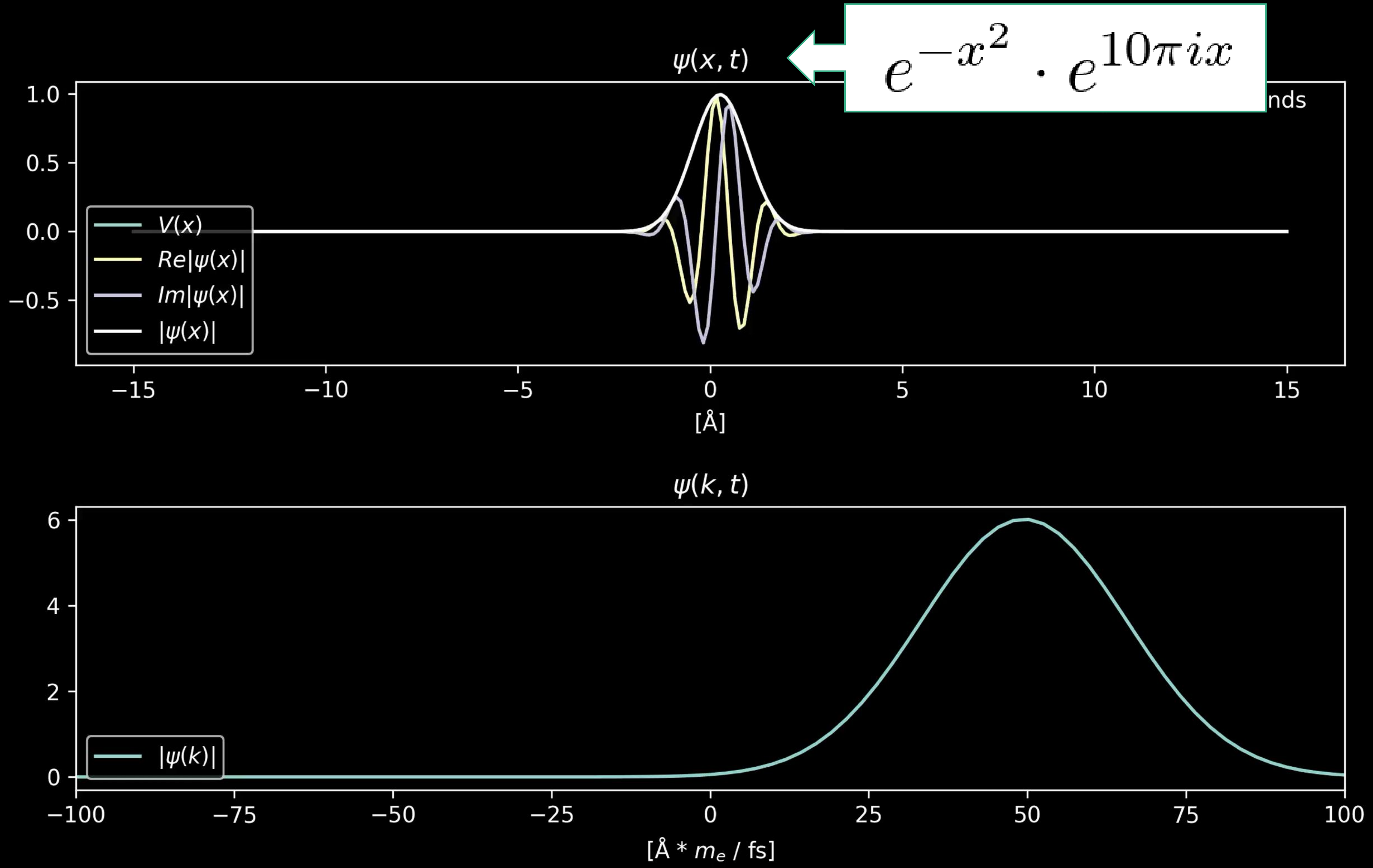
- (it's complicated)
- But here are a few examples:



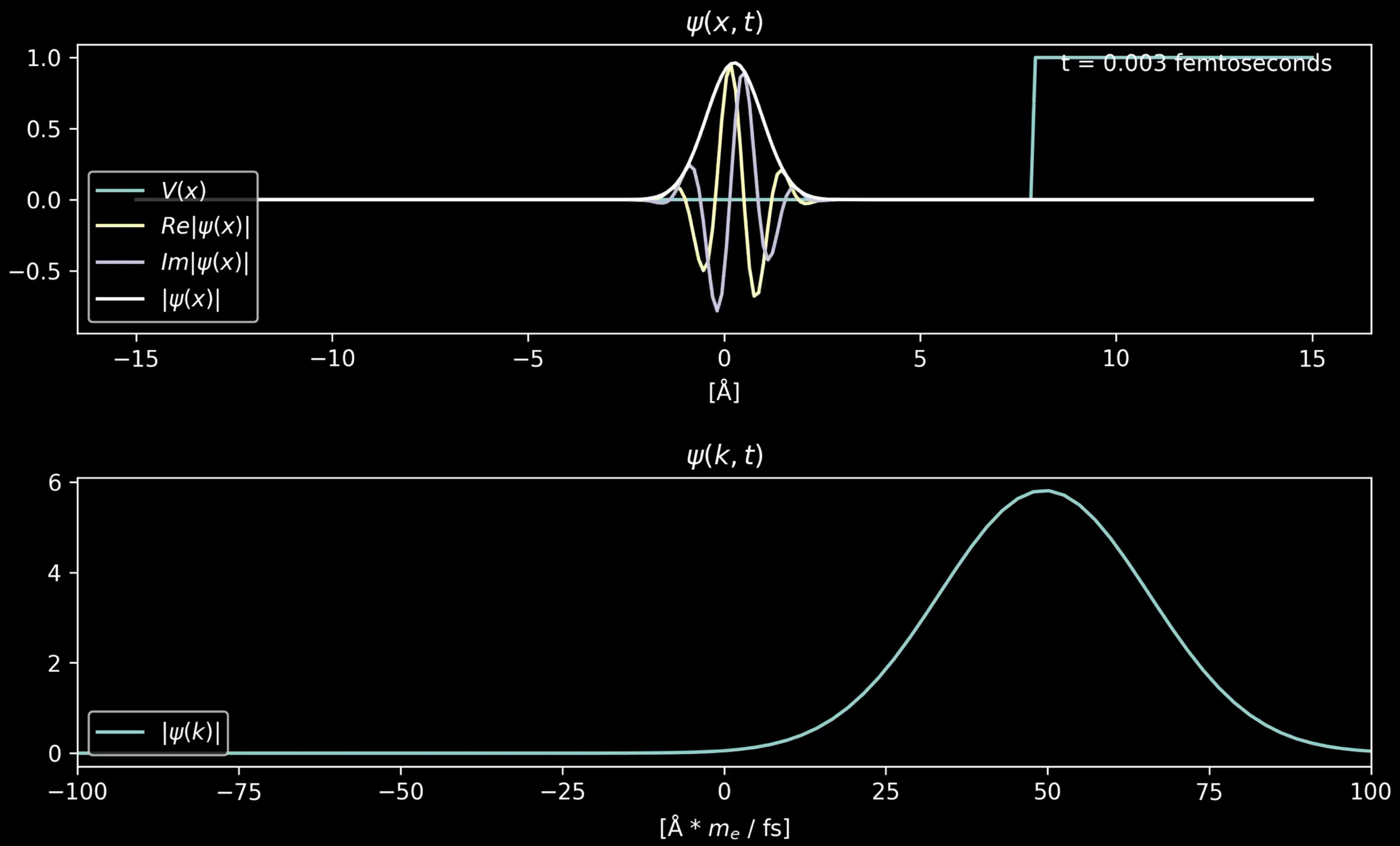
$e^{10\pi ix}$
ψ(x, t)
mtoseconds
1.0
0.5
0.0
−0.5
−1.0
V(x)
Re|ψ(x)|
Im|ψ(x)|
|ψ(x)|
−15
−10
−5
0
5
10
15
[Å]
ψ(k, t)
100
80
60
40
20
0
|ψ(k)|
−100
−75
−50
−25
0
25
50
75
100
[Å * m_e / fs]



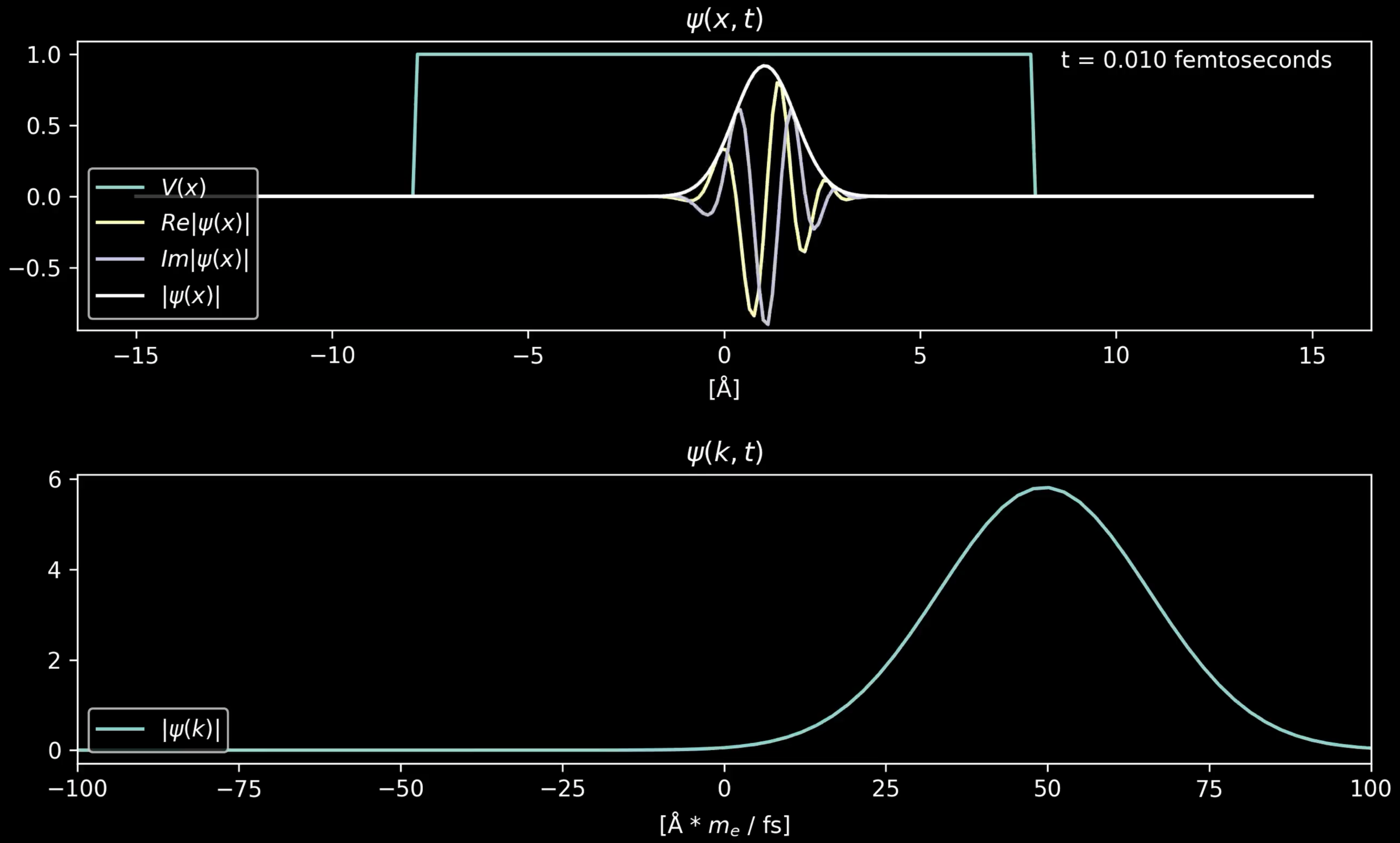
ψ(x, t)
e^{−10πix}
V(x)
Re|ψ(x)|
Im|ψ(x)|
|ψ(x)|
1.0
0.5
0.0
−0.5
−1.0
−15
−10
−5
0
5
10
15
[Å]
ψ(k, t)
100
80
60
40
20
0
|ψ(k)|
−100
−75
−50
−25
0
25
50
75
100
[Å * m_e / fs]



$e^{-x^2} \cdot e^{10\pi i x}$
$\psi(x,t)$
nds
$V(x)$
$Re|\psi(x)|$
$Im|\psi(x)|$
$|\psi(x)|$
1.0
0.5
0.0
−0.5
−15
−10
−5
0
5
10
15
[Å]
$\psi(k,t)$
$|\psi(k)|$
6
4
2
0
−100
−75
−50
−25
0
25
50
75
100
[Å * m_e / fs]



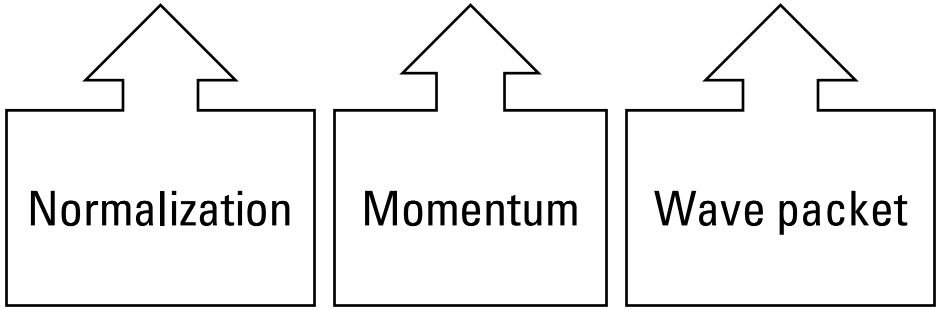
ψ(x, t)
t = 0.003 femtoseconds
V(x)
Re|ψ(x)|
Im|ψ(x)|
|ψ(x)|
1.0
0.5
0.0
−0.5
−15
−10
−5
0
5
10
15
[Å]
ψ(k, t)
|ψ(k)|
6
4
2
0
−100
−75
−50
−25
0
25
50
75
100
[Å * m_e / fs]


ψ(x, t)
t = 0.010 femtoseconds
1.0
0.5
0.0
−0.5
V(x)
Re|ψ(x)|
Im|ψ(x)|
|ψ(x)|
−15
−10
−5
0
5
10
15
[Å]
ψ(k, t)
6
4
2
0
|ψ(k)|
−100
−75
−50
−25
0
25
50
75
100
[Å * m_e / fs]

Proving the Uncertainty Principle

- Start with a wave packet:

$$\frac{1}{\sqrt{2\Delta x}} e^{i\frac{p_0 x}{\hbar}} e^{-\frac{|x|}{2\Delta x}}$$



Proving the Uncertainty Principle

- Start with a wave packet:

$$\frac{1}{\sqrt{2\Delta x}} e^{i\frac{p_0 x}{\hbar}} e^{-\frac{|x|}{2\Delta x}}$$

- And the Fourier transform:

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,t) e^{-\frac{ipx}{\hbar}}\, dx$$

Normalization

"Conversion factor" between wavelength and momentum

Proving the Uncertainty Principle

- Substitute…

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\Delta x}} e^{\frac{ip_0 x}{\hbar}} e^{-\frac{|x|}{2\Delta x}} \right) e^{-\frac{ipx}{\hbar}}\, dx$$

- Split into positive and negative regions and combine coefficients and exponents…

$$\frac{1}{2\sqrt{\pi\hbar\Delta x}} \int_0^{\infty} e^{\left(x\frac{2\Delta x i(p_0-p)-\hbar}{2\Delta x\hbar}\right)}\, dx + \frac{1}{2\sqrt{\pi\hbar\Delta x}} \int_{-\infty}^{0} e^{x\left(\frac{2\Delta x i(p_0-p)+\hbar}{2\Delta x\hbar}\right)}\, dx$$

- Find the integrals…

$$\frac{2\Delta x\hbar}{2\sqrt{\pi\hbar\Delta x}} \left(\left[\frac{1}{2\Delta x i(p_0-p)-\hbar} e^{\frac{x(2\Delta x i(p_0-p)-\hbar)}{2\Delta x\hbar}}\right]_0^{\infty} + \left[\frac{1}{2\Delta x i(p_0-p)+\hbar} e^{\frac{x(2\Delta x i(p_0-p)+\hbar)}{2\Delta x\hbar}}\right]_{-\infty}^{0} \right)$$

Proving the Uncertainty Principle

- Substitute and simplify:

$$\sqrt{\frac{\Delta x\hbar}{\pi}}\left(\frac{1}{\hbar-2\Delta xi(p_0-p)}+\frac{1}{\hbar+2\Delta xi(p_0-p)}\right)$$

$$=\sqrt{\frac{\Delta x\hbar}{\pi}}\left(\frac{2\hbar}{\hbar^2+4\Delta x^2(p_0-p)^2}\right)$$

$$=2\sqrt{\frac{\Delta x}{\hbar\pi}}\left(\frac{1}{1+\left(\frac{2\Delta x}{\hbar}(p_0-p)\right)^2}\right)$$

Proving the Uncertainty Principle

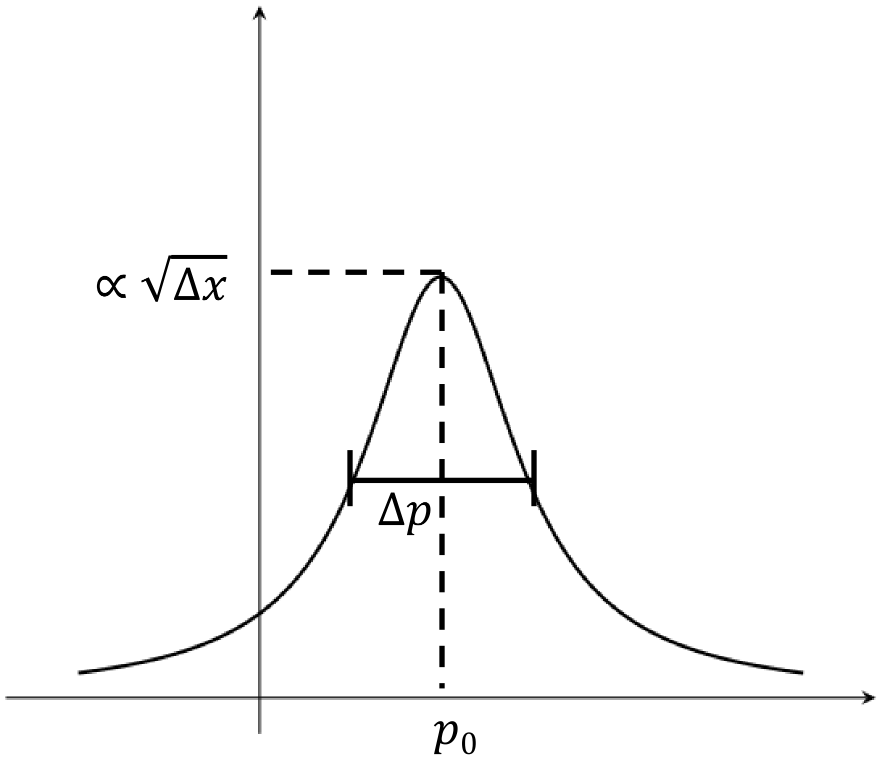
$$2\sqrt{\frac{\Delta x}{\hbar\pi}}\left(\frac{1}{1+\left(\frac{2\Delta x}{\hbar}(p_0-p)\right)^2}\right)$$

- Take a Look:
- Define Δp to be distance between points where magnitude halves and solve:

$$\frac{1}{2}=\frac{1}{1+\frac{2\Delta x}{\hbar}(p_0+\frac{\Delta p}{2}-p_0)^2}$$

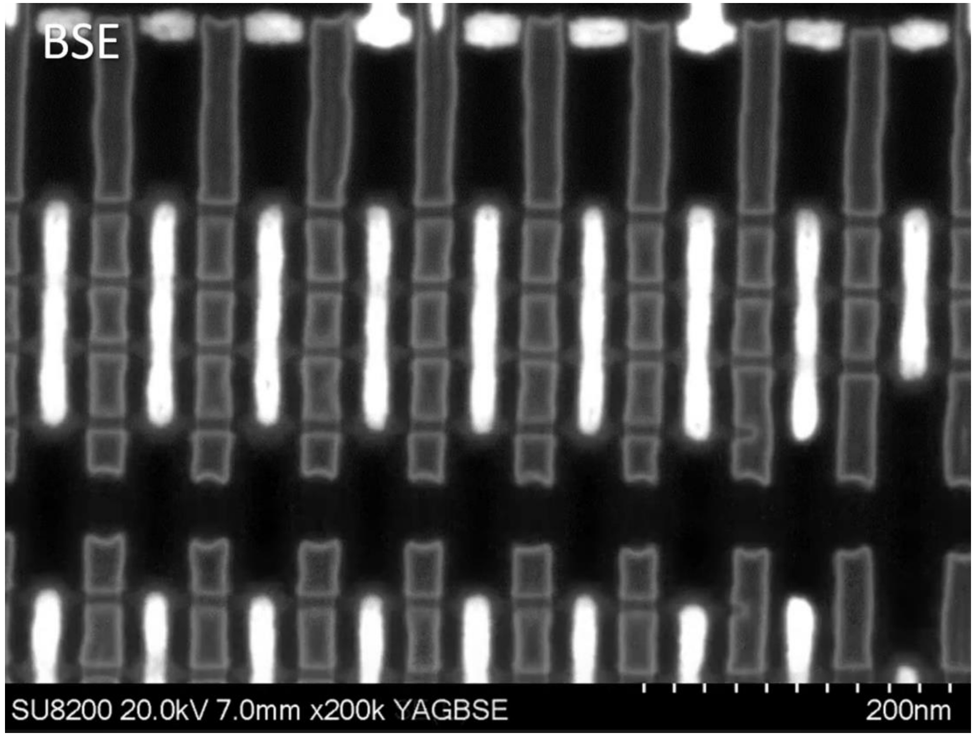
$$1=\frac{2\Delta x}{\hbar}\cdot\frac{\Delta p}{2}$$

$$\hbar=\Delta x\cdot\Delta p$$



Who cares?

- Modern integrated circuits are nanometers in scale
- You want to know where electrons are (small Δx), e.g. to determine whether a bit of memory is 1 or 0
- And you don't want them to move about and change things (small Δp)
- The Heisenberg Uncertainty Principle shows that there is a fundamental limit to which electronics can be minimized



14-nm FinFET SRAM under electron microscope

Thank you!