EE105 Final Project:

Financial Derivative Pricing as a Feedback Control System

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November 29, 2025

Introduction

Financial markets aren't what usually comes to mind when one thinks of a feedback control system, however many of the approaches and mathematical tools used to study mechanical and electrical dynamical systems can also be applied to finance. For my final project, I apply feedback control theory to one area of finance: derivative pricing. Specifically, I use FCS principles to derive the Black-Scholes equation[1], then use the Laplace transform to analytically solve the differential equation for a simpler contract and numerical methods to solve the equation for more complex derivatives, comparing the resulting prices to real markets.

In financial markets, derivatives are contracts whose values are derived (hence "derivative") from the prices of other assets, such as a stock or commodity. Two common derivative products are calls and puts: calls give the option for the owner to buy an asset at a set price ("strike price") by a specific time (expiration data) and puts give the option for the owner to sell an asset at a set price by a specific time. The value of calls and puts at the expiration date are therefore dependent on both the value of the asset and the strike price:

$$C_T = \max(S_T - K, 0) \qquad P_T = \max(K - S_T, 0)$$

The value of a put and call option before the expiration time is nontrivial and set by market forces. Therefore, it is of interest to derive the true value of an option so that one can arbitrage between the true value and market price if they differ, generating profit.

Method

(a) Deriving the Black-Scholes model

One well-known approach for pricing derivatives is the Black-Scholes model. In the Black-Scholes model, the true value of the derivative is computed by modeling a portfolio in which the option is held alongside a number of the underlying asset. The number of assets is determined by the amount that will cancel out changes in the portfolio value when the underlying price changes, causing the portfolio to become riskless. Then, in an efficient market, the total value of the portfolio must grow at the risk-free-interest rate.

This system can be modeled as a two-part control system, where the first part models the underlying asset (such as a stock) and the second part models the portfolio. The diagram of the control system is shown in Figure 1a. In an efficient market, the portfolio should grow at the same rate as a risk-free portfolio, which is shown in Figure 1b.

The stock portion models a system that grows proportionally with a gain factor that consists of a constant term μ and a stochastic term σ. W follows standard Brownian motion and dW/dt corresponds to the change in that signal per unit time. The resulting equation modeling the stock price is then:

$$\frac{dS}{dt} = S\left(\mu + \sigma\frac{dW}{dt}\right).$$

Multiplying by dt on both sides,

$$dS = \mu S\, dt + \sigma S\, dW.$$

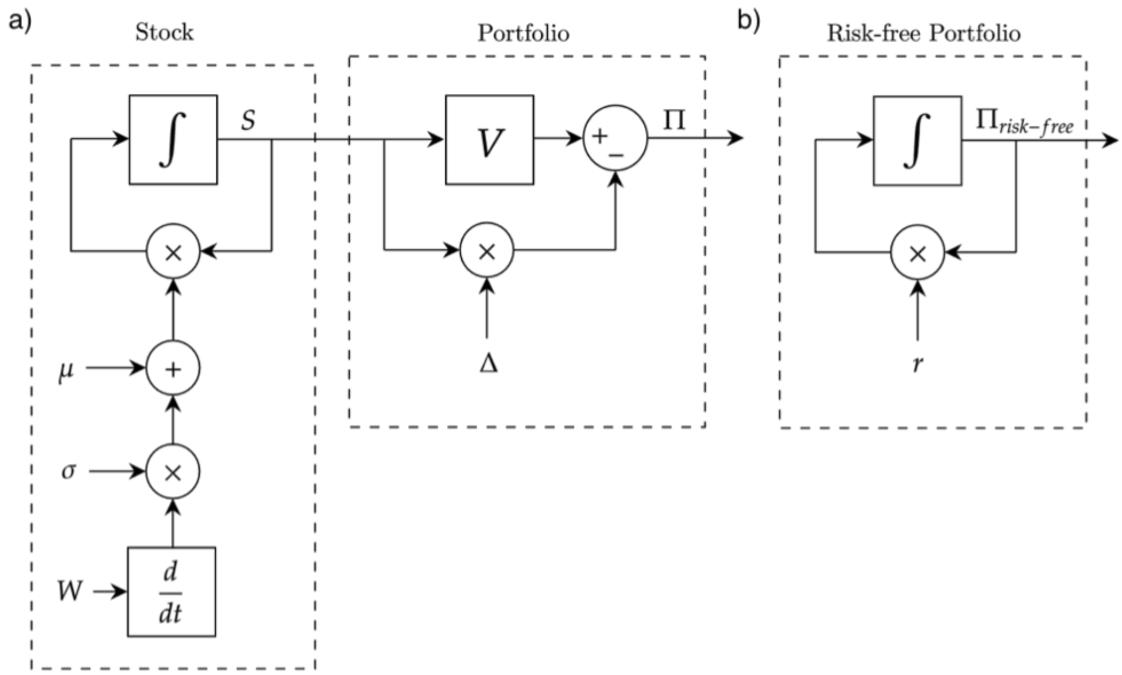

Figure 1: (a) Diagram of the two part-control system, where the first part models the stock and the second part models the portfolio. (b) Diagram of a risk-free portfolio.

Then, to derive dV (the change in the value of the derivative), I apply Ito's lemma[2]. Because W follows Brownian motion, the standard chain rule cannot be applied, as Brownian motion is nowhere differentiable and has unusual analytic properties. Specifically, Ito's rules states that $dW^2 = dt$, $dW\,dt = 0$, and $dt^2 = 0$. Applying Taylor expansion and substituting these terms yields Ito's lemma, which states that for

$$dX = a(X,t)\,dt + b(X,t)\,dW,$$

and $Y = f(X,t)$,

$$dY = \left(\frac{\partial f}{\partial t} + a(X,t)\,\frac{\partial f}{\partial X} + \frac{1}{2}b^2(X,t)\,\frac{\partial^2 f}{\partial X^2}\right) dt + b(X,t)\,\frac{\partial f}{\partial X}\,dW.$$

Applying Ito's Lemma,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\,dS + \frac{1}{2}\,\frac{\partial^2 V}{\partial S^2}\,dS^2.$$

Then,

$$dS^2 = (\mu S\,dt + \sigma S\,dW)^2$$

and since $dt^2 = dt\,dW = 0$ and $dW^2 = dt$,

$$dS^2 = \sigma^2 S^2\,dt.$$

Substituting this,

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW.$$

The portfolio Π consists of a single option V and Δ number of the shorted stock S, so

$$\Pi = V - \Delta S$$

and,

$$d\Pi = dV - \delta\, dS.$$

Substituting the previous equations,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW - \delta \left(\mu S\, dt + \sigma S\, dW \right).$$

To make the portfolio riskness, it is necessary to cancel out the dW term. This appears twice, once in the derivative term and one in the underlying term. By setting $\delta = \partial V / \partial S$, the two terms cancel out, while also canceling out the $\mu S(\partial V / \partial S)$ term:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Since the portfolio is riskless, it must earn at the riskless rate of return:

$$d\Pi = r\Pi\, dt = r(V - \Delta S)\, dt.$$

Substituting,

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left(V - \frac{\partial V}{\partial S} S \right) dt.$$

Rearranging this yields the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

(b) Solving Black-Scholes with the Laplace transform

For many derivatives, like American put and call options, there isn't a closed form solution for the Black-Scholes equation. Here, I derive the closed form solution for a simpler derivative, a step payoff, where the value of the derivative is

$$V(S, t = T) = \begin{cases} 1 & \text{if } S \geq K, \\ 0 & \text{otherwise,} \end{cases}$$

where K is the strike price and T is the time of expiration.

To simplify the derivation, first define the time to expiration τ as $T - t$ and substitute τ

into Black-Scholes, resulting in:

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV.$$

Then, I perform a change of variables into u and z, defining

$$V(S,\tau) = e^{-r\tau}\, u(z,\tau)$$

$$z = \ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau.$$

The derivation of this transformation is out of scope (it is long and convoluted), but by computing derivatives, substituting, and canceling out terms, it can be verified that the equation simplifies to,

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2}$$

which is the heat equation.

The initial condition then becomes

$$u(z,0) = e^{r\cdot 0} V(S, T-T) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. still the step function.

Now, to solve the partial differential equation, I perform a Laplace transform on τ, transforming it into an ordinary differential equation, which can then be solved directly.

The left side becomes,

$$\mathcal{L}\left\{\frac{\partial u}{\partial \tau}\right\} = sU(z,s) - u(z,0)$$

And the right stays,

$$\mathcal{L}\left\{\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial z^2}\right\} = \frac{\sigma^2}{2}\frac{\partial^2 U(z,s)}{\partial z^2}.$$

Now define $\lambda = \sqrt{\frac{2s}{\sigma^2}}$ and $a = \sigma^2/2$, resulting in the Laplace-transformed equation:

$$\frac{\partial^2 U}{\partial z^2} - \lambda^2 U = -\frac{1}{a}\cdot \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

To solve this ODE, consider each part separately. For $z < 0$, the equation simplifies to $\partial^2 U/\partial z^2 = \lambda^2 U$, which has the solution form,

$$U(z,s) = Ae^{\lambda z}.$$

(Ignoring terms that would cause the equation to blow up). Similarly, for $z > 0$, the equation simplifies to $\partial^2 U/\partial z^2 - \lambda^2 U = -1/a$, which has a known solution form

$$U(z,s) = \frac{1}{s} + Be^{-\lambda z}.$$

To solve for A and B, I use the fact that U and it's derivative $\partial U/\partial z$ must be continuous at $z = 0$. For the continuity of U at $z = 0$:

$$Ae^{\lambda \cdot 0} = \frac{1}{s} + Be^{-\lambda \cdot 0} \Rightarrow A = \frac{1}{s} + B$$

For the continuity of $\partial U/\partial z$ at $z = 0$:

$$\frac{\partial}{\partial z}\left(Ae^{\lambda z}\right)\bigg|_{z=0} = \frac{\partial}{\partial z}\left(\frac{1}{s} + Be^{-\lambda z}\right)\bigg|_{z=0}$$

$$A\lambda = -B\lambda$$

$$A = -B$$

Solving the pair results in $A = 1/2s$ and $B = -1/2s$, and gives the Laplace-space solution:

$$U(z,s) = \begin{cases} \frac{1}{s}e^{\lambda z} & \text{for } z < 0, \\ \frac{1}{s} - \frac{1}{2s}e^{-\lambda z} & \text{for } z > 0, \end{cases}$$

Substituting $\lambda = \sqrt{s/a}$,

$$U(z,s) = \begin{cases} \frac{1}{s}e^{\sqrt{s/a}\,z} & \text{for } z < 0, \\ \frac{1}{s} - \frac{1}{2s}e^{-\sqrt{s/a}\,z} & \text{for } z > 0. \end{cases}$$

Finally, I found the inverse transform using the standard pairs:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-c\sqrt{s}}\right\} = \text{erfc}\left(\frac{c}{2\sqrt{\tau}}\right) \quad \text{(where erfc is the complementary error function)}$$

Which results in

$$u(z,\tau) = \begin{cases} \frac{1}{2}\,\text{erfc}\left(\frac{-z}{2\sqrt{a\tau}}\right) & \text{for } z < 0, \\ 1 - \frac{1}{2}\,\text{erfc}\left(\frac{z}{2\sqrt{a\tau}}\right) & \text{for } z > 0. \end{cases}$$

By substituting $\text{erf}(x) = 1 - \text{erfc}(x)$ and acknowledging that erf is odd, both sides simplify to:

$$u(z,\tau) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{z}{2\sqrt{a\tau}}\right)\right)$$

And substituting $a = \sigma^2/2a$ gives:

$$u(z,\tau) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{z}{\sigma\sqrt{2\tau}}\right)\right)$$

For further simplicity, note that the normal CDF is defined as $N(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$ so u

simplifies to:

$$u(z, \tau) = N\left(\frac{z}{\sigma\sqrt{\tau}}\right)$$

I lastly undo the change-of-variables to get:

$$V(S, \tau) = e^{-r\tau} N\left(\frac{\ln(S/K) + (r - \frac{1}{2}\omega^2)\tau}{\sigma\sqrt{\tau}}\right)$$

and since $\tau = T - t$,

$$V(S, t) = e^{-r(T-t)} N\left(\frac{\ln(S/K) + (r - \frac{1}{2}\omega^2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

This is the solution of the Black–Scholes equation for the step payoff derivative.

Results

Next, I'll apply the equations derived above to pricing both simulated markets and real markets, comparing the computed prices to the actual fair prices.

(a) Evaluating the step payoff Black-Scholes solution

For the step-payoff derivative, it is necessary to use a simulated market to derive fair pricing as it isn't a commonly traded contract. To construct the simulated stock market, I defined the paths that the stock prices would take as geometric Brownian motion. Specifically, the price at each time step is derived as follows,

$$S_{t+\delta t} = S_t \exp\left[(r - \tfrac{1}{2}\sigma^2)\,\delta t + \sigma\sqrt{\delta t}\,Z\right],$$

where S_t is the price of the stock at time t, r is the expected rate of return, σ is volatility (the degree that the price changes randomly), δt is the time step, and Z is sampled from the normal distribution: $Z \sim N(0, 1)$.

Figure 2 shows a simulation of 10 stock trajectories, each starting at price 100, growing at 5% with volatility 20%. Interestingly, even though the stock trajectories are completely random and constructed through an extremely simple model, they still resemble the stock price graphs observed in real markets, with random hills and valleys, sudden spikes and drops, and plateaus.

I'll next plot the step payoff Black-Scholes solution to understand how each parameter affects the resulting prices. Figure 3 shows the plot of the solved prices for various parameter settings.

In general, the form of the plots is intuitive. At high stock prices, the price of the option is close to 1, as it is unlikely that the stock would dip to below the strike price. Similarly, at low prices, the price of the option is close to 0, as it is unlikely that the stock would rise to above the strike price before expiration. At expiration, the price of the option is a step function, matching the payoff exactly. As the time to expiration increases, the discrete step becomes a smooth sigmoidal curve as it becomes more likely for the stock to pass through the strike price before expiration.

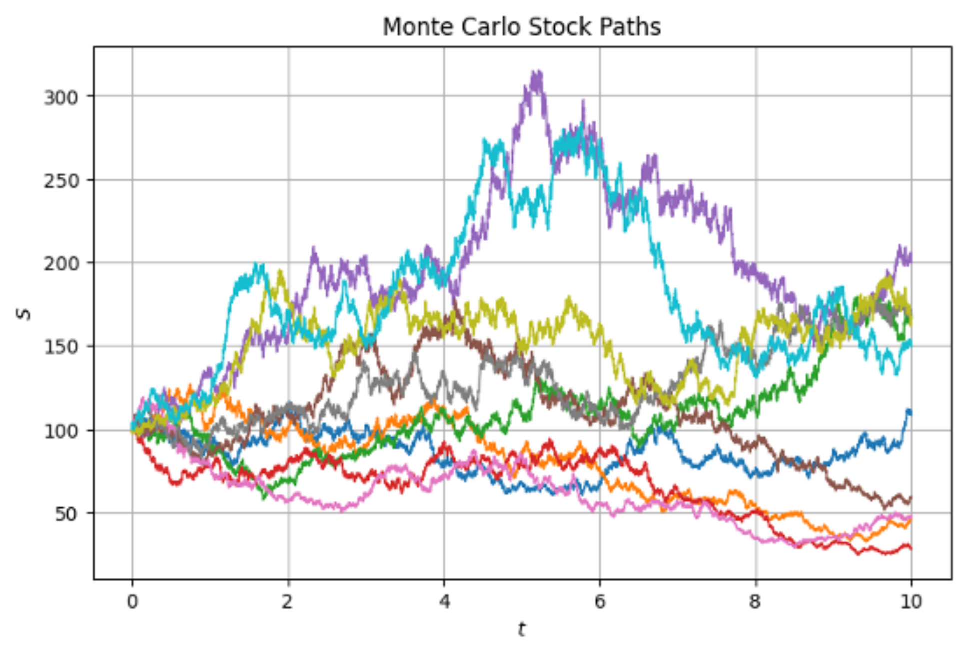

Figure 2: Simulations of 10 stock trajectories with $S_0 = 100$, $r = 0.05$, and $\sigma = 0.2$ up to time 10 using 10,000 individual time steps.

Figure 3b shows the solution with the same parameters as 3a but with a higher risk-free return. For times close to expiration, the prices are similar to 3a, but as time to expiration increases, the price of the option drops exponentially compared to the corresponding time and stock price in 3a. This is because the present-time prices must be discounted according to the risk-free return. Suppose if an option was sold for 1$ and the seller invested the 1 dollar in the risk-free asset. Then after 10 time units with 10% growth that one dollar would become $e^{0.1 \cdot 10} = \$2.71$, making it trivially easy to cover the option if it paid out. Therefore, the price of the option must be discounted according to the risk-free rate.

Figure 3c shows the shows the solution with the same parameters as 3a but with a higher volatility. As exacted, the rate that the step payoff becomes a smooth sigmoidal shape increases compared to 3a because it is more likely for the stock to cross the strike price due to volatility. However, there is also an additional effect where long time-to-expiration options are discounted more than they are in 3a. This is due to the same non-intuitive effect shown in 3d.

Figure 3d shows the shows the solution with the same parameters as 3a but with a zero risk-free return rate. Intuitively, the price of the option should remain symmetric around the strike price as the expected value of the stock remains constant due to the zero risk-free return. However, for long time-to-expiration cases, the price of the option actually gets a discount. The reason this occurs is due to the fact that the stock price follows geometric Brownian motion rather than 'normal' Brownian motion. As geometric Brownian motion evolves, even when the mean stays constant, the distribution becomes extremely skewed, with a few samples at high values "canceling out" many samples at low values. Therefore, with long time-to-expiration, with a stock at the strike price there is still a greater change for the stock to end below the strike price at expiration (as it is more likely to one of the samples forming the bottom of the distribution) even when the expected value of the stock at expiration is equal to the strike price,

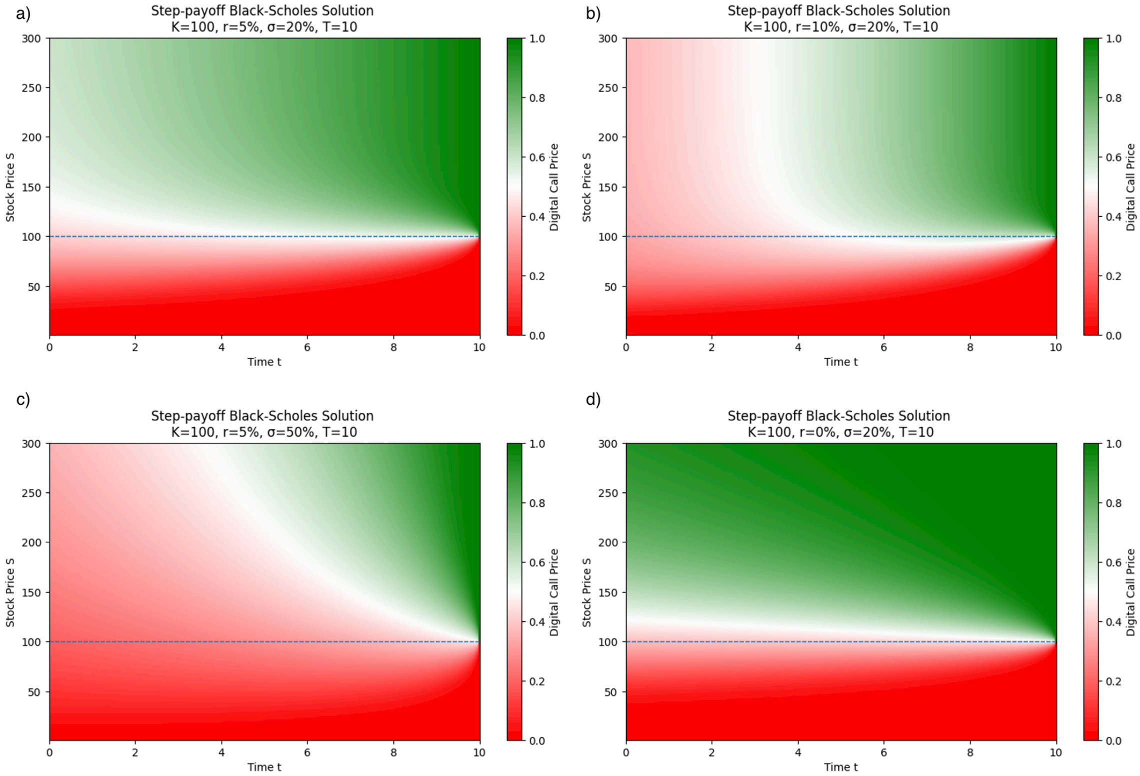

Figure 3: Step-payoff black-scholes solution plots for various parameters over time period 10 and strike price 100. (a) $r = 5\%$, $\sigma = 20\%$. (b) $r = 10\%$, $\sigma = 20\%$. (c) $r = 5\%$, $\sigma = 50\%$. (d) $r = 0\%$, $\sigma = 20\%$.

causing the option to be discounted.

Lastly, to show correctness of the Black-Scholes solution, I performed a simulation of 1000 stock trajectories starting from a grid of price and time values to compare to the analytical solution, where the value shown at each point is the average of the payoffs of each simulation. The parameters were $r = 5\%$ and $\sigma = 20\%$ with a grid density of 20x20. As expected, aside from noise, both plots look identical.

(b) Solving Black-Scholes for real market pricing

Next, I'll apply Black-Scholes to pricing options on the real market. Specifically, I'll use the closed form solution of Black-Scholes for European call options. The derivation of this solution is identical to the solution for the step payoff but with a different boundary condition, which complicates the solution somewhat. The closed form solution, given the underlying price S, strike price K, time to expiration T, risk free rate r, and volatility σ, is:

$$V(S,T) = S\,N(u) - K\,e^{-rT}N(u - \sigma\sqrt{T})$$

Where

$$u = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and N is the normal cdf.

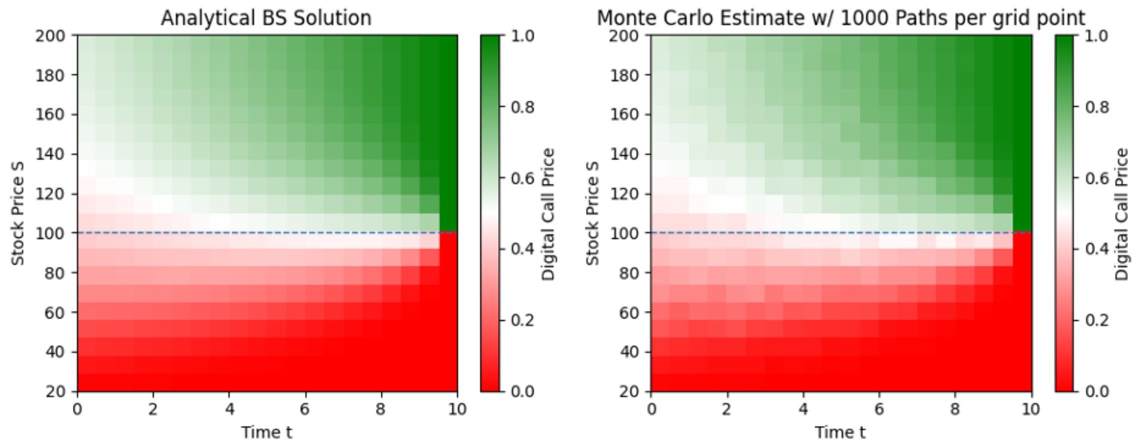

Figure 4: Analytical solution vs. Monte-Carlo Estimate of price of option. Each grid point is the average of 1000 simulations with were $r = 5\%$ and $\sigma = 20\%$.

I collected real time data for the S&P500 (SPX), Nasdaq (NDX), and Apple, Inc. (AAPL) call prices over different strikes and expirations. These plots are shown in the first column of Figure 5. Note that in these plots, the y axis is the strike price, not the underlying price, which is why the cost is inverted (higher strike = cheaper) compared to previous plots. Then, I fitted least square values for r and σ and plotted the resulting price map on the second column of Figure 5.

Overall, the fitted values are extremely close to the actual market prices, indicating that the solution models the market well. Trends such as the expected return and the smoothing of prices as the forecasted period increases are modeled. Additionally, the fitted values of r and σ are sensible. For the S&P 500, the return is 4.7% (year over year) with a volatility of 10.64%, while the Nasdaq, which consists of fewer companies that are more tech focused, has a higher return at 10% albeit at a higher volatility of 15.88%. Apple, an individual high-cap company in both indexes has a similar 10% expected return but at a higher 22.93% volatility.

Conclusion

In my final project, I applied feedback control theory to price derivatives in finance. I derived the Black-Scholes equation as an FCS system, then use the Laplace transform to analytically solve the differential equation for a step payoff contract. I verified the analytical solution for the step payoff via monto-carlo simulations and demonstrated that the resulting prices are sensible. Lastly, I numerically fit the solution for European call contracts to real market data and showed that the model fits the real prices extremely well. These results show how FCS principles can be applied outside of mechanical and electrical systems towards other fields like finance.

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1973.

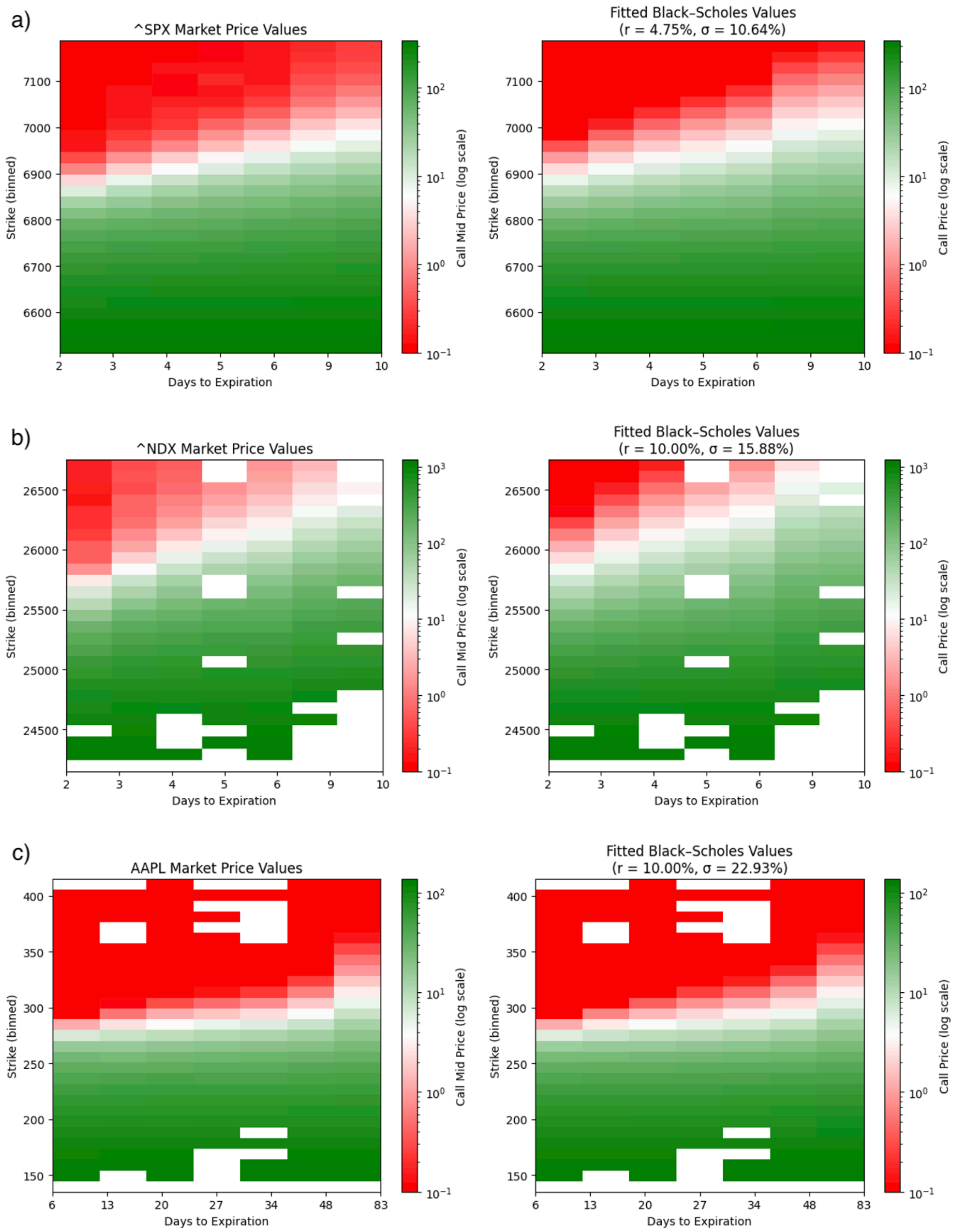

Figure 5: Market call prices and computed call prices using least-square fit for r and σ. Tickers: (a) S&P 500 / SPX (b) Nasdaq / NDX (c) Apple, Inc / AAPL.

[2] K. Itō, "Stochastic integral," *Proceedings of the Imperial Academy*, vol. 20, no. 8, pp. 519–524, 1944.